

Geometry I, 2006 : Coursework 4 Solutions

[Note: It is obvious that applying elementary operation (iii) (interchanging two equations) does not change the set of solutions of a system of linear equations in x, y, z . Exercise 1 shows that applying elementary operation (ii) does not change this set, and Exercise 2 shows that applying elementary operation (i) also does not change this set.]

1. (i) [2 marks] Suppose $x = p, y = q, z = r$ is a solution of $ax + by + cz = d$; that is, $ap + bq + cr = d$. Then $\alpha ap + \alpha bq + \alpha cr = \alpha(ap + bq + cr) = \alpha d$, and so $x = p, y = q, z = r$ is a solution of $\alpha ax + \alpha by + \alpha cz = d$.

(ii) [2 marks] Conversely, suppose $x = p, y = q, z = r$ is a solution of $\alpha ax + \alpha by + \alpha cz = \alpha d$. Since $\alpha \neq 0$, we have $ap + bq + cr = \frac{1}{\alpha}(\alpha ap + \alpha bq + \alpha cr) = \frac{1}{\alpha}(\alpha d) = d$, and so $x = p, y = q, z = r$ is a solution of $ax + by + cz = d$.

2. (i) [2 marks] Suppose $x = p, y = q, z = r$ is a solution of (*). Then $ap + bq + cr = d$. What about the second equation of (**)? Well,

$$(e + \alpha a)p + (f + \alpha b)q + (g + \alpha c)r = (ep + fq + gr) + \alpha(ap + bq + cr) = h + \alpha d,$$

and so $x = p, y = q, z = r$ is a solution to the second equation of (**) as well as the first, and we are done.

(ii) [2 marks] Conversely, suppose $x = p, y = q, z = r$ is a solution of (**). Then $ap + bq + cr = d$. What about the second equation of (*)? Well,

$$ep + fq + gr = (e + \alpha a)p + (f + \alpha b)q + (g + \alpha c)r - \alpha(ap + bq + cr) = (h + \alpha d) - \alpha d = h,$$

and so $x = p, y = q, z = r$ is a solution to the second equation of (*) as well as the first, and we are done.

3. [2 marks each part; 1 mark for reasonably correct calculations or explanation, and 1 mark for all solutions to the system of equations. No marks for answer only.]

$$(i) \quad \begin{cases} 3x & - & y & - & 2z & = & -11 \\ x & - & y & + & z & = & -6 \\ 5x & - & 3y & - & 4z & = & -23 \end{cases}$$

Interchange first and second equations:

$$\begin{cases} x & - & y & + & z & = & -6 \\ 3x & - & y & - & 2z & = & -11 \\ 5x & - & 3y & - & 4z & = & -23 \end{cases}$$

Add -3 times first equation to second, and -5 times first to third:

$$\begin{cases} x & - & y & + & z & = & -6 \\ & & 2y & - & 5z & = & 7 \\ & & 2y & - & 9z & = & 7 \end{cases}$$

Add -1 times second equation to third:

$$\begin{cases} x & - & y & + & z & = & -6 \\ & & 2y & - & 5z & = & 7 \\ & & & & -4z & = & 0 \end{cases}$$

The system is now in echelon form. We have $z = 0$, $2y - 0 = 7$, so $y = 7/2$, and $x = -6 + 7/2 - 0 = -5/2$.

The only solution to the system is $x = -5/2, y = 7/2, z = 0$.

[Alternatively, the set of solutions is $\{(-5/2, 7/2, 0)\}$ (but **not** $\{-5/2, 7/2, 0\}$).]

$$(ii) \quad \begin{cases} 3x - y - 2z = 11 \\ 6x - y - 4z = 22 \end{cases}$$

Add -2 times the first equation to the second:

$$\begin{cases} 3x - y - 2z = 11 \\ y = 0 \end{cases}$$

The system is now in echelon form. We have that z can take any real value, say $z = t$, $y = 0$, and then $x = (11 + 2t)/3$.

[Alternatively, the set of solutions is $\{((11 + 2t)/3, 0, t) : t \in \mathbb{R}\}$.]

$$(iii) \quad \begin{cases} 243x + 53y + 163z = 912 \\ 1024x + 28y + 923z = 78 \\ 1024x + 28y + 923z = 79 \\ 847x - 33y + 7843z = -105 \end{cases}$$

Adding -1 times the second equation to the third, we obtain $0 = 1$.

Hence this system has no solutions.

[Alternatively, the set of solutions is $\emptyset$, the empty set.]

4. (i) The intersection is the set of solutions of the following system of linear equations:

$$\begin{cases} 2x - 3y + 6z = 4 \\ -4x + 6y - 12z = -8 \end{cases}.$$

Adding 2 times the first equation to the second, we get

$$\begin{cases} 2x - 3y + 6z = 4 \\ 0 = 0 \end{cases}.$$

The system is now in echelon form. We have that y and z can take any real values, say $y = s$ and $z = t$, and then $x = (4 + 3s - 6t)/2 = 3s/2 - 3t + 2$.

[1 mark for the intersection; for this mark, the intersection must be specified correctly as a set of points (or simply as Π).]

The intersection is thus

$$\{(3s/2 - 3t + 2, s, t) : s, t \in \mathbb{R}\}.$$

[Alternatively: the intersection is Π .]

[1 mark for a correct equation for the plane]

This intersection is the plane Π , which has equation $2x - 3y + 6z = 4$.

(ii) [1 mark for correct solving of system] The intersection is the set of solutions of the following system of linear equations:

$$\begin{cases} 2x - 3y + 6z = 4 \\ -4x + 6y - 12z = 8 \end{cases}.$$

Adding 2 times the first equation to the second, we get

$$\begin{cases} 2x & - & 3y & + & 6z & = & 4 \\ & & & & 0 & = & 16 \end{cases}.$$

There is no solution to the system.

[1 mark for correct intersection] The intersection is $\emptyset$, the empty set.

(iii) The intersection is the set of solutions of the following system of linear equations:

$$\begin{cases} 2x & - & 3y & + & 6z & = & 4 \\ -4x & + & 6y & - & 11z & = & 8 \end{cases}.$$

Adding 2 times the first equation to the second, we get

$$\begin{cases} 2x & - & 3y & + & 6z & = & 4 \\ & & & & z & = & 16 \end{cases}.$$

The system is now in echelon form. We have that $z = 16$, y can take any real value, say $y = s$, and then $x = (4 + 3s - 96)/2 = 3s/2 - 46$.

[1 mark for the intersection, which must be given as a set of points] The intersection is thus

$$\{(3s/2 - 46, s, 16) : s \in \mathbb{R}\}.$$

[1 mark for correct equations for this line] This intersection is a line, having parametric equations:

$$\begin{cases} x & = & -46 & + & \frac{3}{2}\lambda \\ y & = & \lambda \\ z & = & 16 \end{cases}.$$

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